

Fluctuations and defect-defect correlations in the ordering kinetics of the O(2) model

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The theory of phase ordering kinetics for the O(2) model using the Gaussian auxiliary field approach is reexamined from two points of view. The effects of fluctuations about the ordering field are included and we organize the theory such that the auxiliary field correlation function is analytic in the short-scaled distance (x) expansion. These two points are connected and we find in the refined theory that the divergence at the origin in the defect-defect correlation function $\bar{g}(x)$ obtained in the original theory is removed. Modifications to the order-parameter autocorrelation exponent λ are computed. [S1063-651X(97)07001-3]

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I. INTRODUCTION

The phase-ordering kinetics of systems with continuous symmetry, such as the O(n) model, is particularly interesting because of the topological defect structures produced: vortices and strings for $n=2$ and monopoles for $n=3$ [1]. While much is understood about the theory of growth kinetics for the O(n) model, there are some interesting unresolved problems associated with the short-distance behavior of the defect-defect correlation function $\bar{g}(x)$. In the theory developed in [2] one expresses the order parameter $\bar{\psi}(\mathbf{R}, t)$ as a local nonlinear function of an auxiliary field $\vec{m}(\mathbf{R}, t)$ which is physically interpreted as the distance, at time t , from position \mathbf{R} to the closest defect. One of the physical motivations for introducing $\vec{m}(\mathbf{R}, t)$ is that it is *smoother* than the order-parameter field. Sharp interfaces or well defined defects produce a nonanalytic structure in the order-parameter scaling function $\mathcal{F}(x)$ at small-scaled distances x which, physically, is responsible for the Porod's law decay seen scattering experiments [3]. The expectation, however, is that the auxiliary field correlation function $f(x)$ will be analytic in this same distance range. In the case of a scalar order-parameter these expectations are supported by theory [2]. However for $n > 1$, as pointed out in [4], this is not the case. One finds a weak nonanalytic component in f and, more significantly, for $n=2$ one can trace this nonanalytic component to an unphysical divergence in $\bar{g}(x)$ at small x [5]. This divergence is not seen in simulations [6] or experiments [7] for $n=d=2$, where $\bar{g}(x)$ apparently approaches zero at the origin.

In this paper we focus on the case $n=2$ and show how these problems can be resolved by taking seriously the assumption that the correlations of the auxiliary field are indeed smoother than those of the order parameter. We find that it is possible to rearrange the theory such that f is analytic in x if we extend the theory to include fluctuations about the ordering field and treat the separation between the ordering field and the fluctuation fields carefully. This is accomplished by introducing a new field $\vec{\Theta}$ which is constructed to ensure that the fluctuations are small, while at the same time compensating for the nonanalyticities in f . It is important to note that we work at zero temperature, so the fluctuations are not thermally driven [8]. Rather, we will see that the correlations in the fluctuations are slaved to the cor-

relations in the order parameter. The theory of Ohta, Jasnow, and Kawasaki (OJK) [9] basically avoids this entire discussion by simply assuming that f has a Gaussian form [10]. There is no self-consistent determination of f in that theory.

It is well established that for late times following a quench from the disordered to the ordered phase the dynamics obey scaling and the system can be described in terms of a single growing length $L(t)$, which is characteristic of the spacing between defects. In this scaling regime the order-parameter correlation function

$$C(12) \equiv \langle \bar{\psi}(1) \cdot \bar{\psi}(2) \rangle \quad (1.1)$$

has a universal equal-time scaling form

$$C(12) = \psi_0^2 \mathcal{F}(x), \quad (1.2)$$

where ψ_0 is the magnitude $\psi = |\bar{\psi}|$ of the order-parameter in the ordered phase. Here we use the shorthand notation where 1 denotes (\mathbf{R}_1, t_1) , and define the scaled length x as $x = R/L(t)$ with $R \equiv |\mathbf{R}| \equiv |\mathbf{R}_2 - \mathbf{R}_1|$. It is also well established that, in the scaling regime, $L(t) \sim t^\phi$, where t is the time after the quench. For the nonconserved models considered here the exponent $\phi = 1/2$. Another measurable quantity is the exponent λ governing the decay of order-parameter autocorrelations, and defined by

$$C(\mathbf{0}, t, t') \sim \frac{1}{L^\lambda(t)} \quad \text{for } t \gg t'. \quad (1.3)$$

This nontrivial exponent can be computed theoretically, along with the scaling function \mathcal{F} [4]. The predictions for λ are in excellent agreement with simulation results [11,12]. The theoretical predictions for \mathcal{F} are also in good agreement with simulations [2].

The dynamics of the defect structures themselves is amenable to theoretical treatment [5]. In this paper we shall mainly be interested in the case $n=d$, where the defects are points. For $n=d-1$ the defects are strings, but the analysis follows closely that for point defects and yields qualitatively similar results. The density of point defects, for $n=d$, is defined as

$$\rho(1) = \sum_{\alpha} q_{\alpha} \delta(\mathbf{R}_1 - \mathbf{x}_{\alpha}(t_1)), \quad (1.4)$$

where $\mathbf{x}_{\alpha}(t_1)$ is the position at time t_1 of the α th point defect, which has a charge q_{α} . Correlations in ρ ,

$$G(12) \equiv \langle \rho(1)\rho(2) \rangle, \quad (1.5)$$

at equal times $t_1 = t_2 = t$ can be shown [5] to decompose into two parts

$$G(\mathbf{R}, t) = n_0(t) \delta(\mathbf{R}) + g(\mathbf{R}, t). \quad (1.6)$$

The first term $n_0(t)$ represents defect self-correlations and is just the total unsigned number density of defects at time t . We will be primarily concerned here with the second term $g(\mathbf{R}, t)$ which measures the correlations between different defects. In the scaling regime it can be shown [5] that $n_0(t) \sim L^{-n}(t)$ and that $g(\mathbf{R}, t)$ has the form

$$g(\mathbf{R}, t) = \frac{1}{L^{2n}(t)} \tilde{g}(x), \quad (1.7)$$

where $\tilde{g}(x)$ is a universal scaling function.

In Sec. II we present the $O(n)$ model and describe the mapping between the auxiliary field and the order parameter. In Sec. III we discuss the separation of the equation of motion into an equation for the evolution of the ordering field and an equation for the dynamics of the fluctuations. The main analytical results of the paper are presented in Sec. IV, where we discuss how the quantities \mathcal{F} , λ , and \tilde{g} are determined through the solution of a nonlinear eigenvalue problem. In Sec. V we calculate the correlations in the fluctuations, assuming that the fluctuation field \vec{u} and the auxiliary field \vec{m} form a set of coupled Gaussian variables. Our numerical analysis of the new nonlinear eigenvalue problem is presented in Sec. VI and the results are discussed in the final section, which also addresses more general issues that indicate directions for future research.

II. MODEL

We consider the $O(n)$ model, which describes the dynamics of a nonconserved, n -component order-parameter field $\vec{\psi}(1) = (\psi_1(1), \dots, \psi_n(1))$. To begin we will work with general n ; however, later we will focus on the interesting case $n=2$. As in previous work in this area [4], the dynamics are modeled using a time-dependent Ginzburg-Landau equation

$$\frac{\partial \vec{\psi}}{\partial t} = -\Gamma \frac{\delta F[\vec{\psi}]}{\delta \vec{\psi}}. \quad (2.1)$$

We assume that the quench is to zero temperature where the usual noise term on the right-hand side is zero [13]. Γ is a kinetic coefficient and $F[\vec{\psi}]$ is the free energy, assumed to be of the form

$$F[\vec{\psi}] = \int d^d r \left(\frac{c}{2} |\nabla \vec{\psi}|^2 + V[\vec{\psi}] \right), \quad (2.2)$$

where the potential $V[\vec{\psi}]$ is chosen to have $O(n)$ symmetry and a degenerate ground state with $\vec{\psi} = \psi_0$. Since only these properties of V will be important in what follows we need not be more specific in our choice for V [14]. With a suitable redefinition of the time and space scales the coefficients Γ and c can be set to one and Eq. (2.1) can be written as

$$\frac{\partial \vec{\psi}}{\partial t} = \nabla^2 \vec{\psi} - \frac{\partial V[\vec{\psi}]}{\partial \vec{\psi}}. \quad (2.3)$$

It is believed that our final results are independent of the exact nature of the initial state, provided it is a disordered state.

The evolution induced by Eq. (2.3) causes $\vec{\psi}$ to order and assume a distribution that is far from Gaussian. It is by now standard to introduce a mapping between the physical field $\vec{\psi}$ and an auxiliary field \vec{m} with more tractable statistics. We can decompose $\vec{\psi}$ exactly as

$$\vec{\psi} = \vec{\sigma}[\vec{m}] + \vec{u}. \quad (2.4)$$

The utility of this decomposition lies in our ability to create a consistent theory with the mapping $\vec{\sigma}$ chosen to reflect the defect structure in the problem, and \vec{u} constructed to be small at late times. Thus \vec{u} represents fluctuations about the ordering field $\vec{\sigma}$. The precise statistics of the fields \vec{m} and \vec{u} will be specified below.

The defect structure [15] is naturally incorporated by using the Euler-Lagrange equation for the order-parameter around a static defect in equilibrium,

$$\nabla_m^2 \vec{\sigma}[\vec{m}] = \frac{\partial V[\vec{\sigma}]}{\partial \vec{\sigma}}, \quad (2.5)$$

to determine the functional dependence of $\vec{\sigma}$ on \vec{m} . The defects are then the nonuniform solutions of Eq. (2.5) which match on to the uniform solution at infinity. Since we expect only the lowest-energy defects, having unit topological charge, will survive to late times the relevant solutions to Eq. (2.5) will be of the form

$$\vec{\sigma}[\vec{m}] = A(m) \hat{m}, \quad (2.6)$$

where $m = |\vec{m}|$ and $\hat{m} = \vec{m}/m$. Thus the interpretation of \vec{m} is that its magnitude represents the distance away from a defect core and its orientation indicates the direction to the defect core. We expect m , away from the defect cores, to grow as L in the late-time scaling regime. Inserting Eq. (2.6) into Eq. (2.5) gives an equation for A ,

$$\nabla_m^2 A - \frac{n-1}{m^2} A - V'[A] = 0, \quad (2.7)$$

where the prime indicates a derivative with respect to A . The boundary conditions are $A(0) = 0$, $A(\infty) = \psi_0$. An analysis of Eq. (2.7) for $n > 1$ and large m yields

$$A(m) = \psi_0 \left[1 - \frac{\kappa}{m^2} + \dots \right], \quad (2.8)$$

where $\kappa = (n-1)/V''[\psi_0] > 0$. The algebraic relaxation of the order parameter to its ordered value is a distinct feature of the $O(n)$ model for $n > 1$. In the scalar case ($n=1$) ψ relaxes exponentially to ψ_0 away from the defects.

We shall also be interested in the stability matrix defined by

$$W_{ij}[\vec{\sigma}] \equiv \frac{\partial^2 V[\vec{\sigma}]}{\partial \sigma_i \partial \sigma_j} = V''[A] \hat{\sigma}_i \hat{\sigma}_j + \frac{V'[A]}{A} (\delta_{ij} - \hat{\sigma}_i \hat{\sigma}_j). \quad (2.9)$$

By the definition of ψ_0 and what we mean by equilibrium we have

$$V'[\psi_0] = 0 \quad (2.10)$$

and

$$V''[\psi_0] \equiv q_0^2 > 0. \quad (2.11)$$

These results give

$$W_{ij}[\psi_0] = q_0^2 \hat{\sigma}_i \hat{\sigma}_j, \quad (2.12)$$

which is purely longitudinal. This reflects the fact that, in equilibrium, the longitudinal fluctuations have a ‘‘mass’’ q_0^2 while the transverse fluctuations, or spin waves, are massless.

III. SEPARATION OF EQUATIONS OF MOTION

In this section we develop the equations of motion satisfied by the fields $\vec{\sigma}$ and \vec{u} . Let us first define

$$H_i[\vec{\psi}] \equiv \frac{\partial V[\vec{\psi}]}{\partial \psi_i} \quad (3.1)$$

and rewrite the equation of motion (2.3) in the form

$$\frac{\partial}{\partial t} (\vec{\sigma} + \vec{u}) = \nabla^2 (\vec{\sigma} + \vec{u}) - \vec{H}[\vec{\sigma} + \vec{u}]. \quad (3.2)$$

We then quite generally assume that $\vec{\sigma}$ satisfies the equation of motion

$$\frac{\partial \vec{\sigma}}{\partial t} = \nabla^2 \vec{\sigma} - \nabla_m^2 \vec{\sigma} + \vec{\Theta}, \quad (3.3)$$

where $\vec{\Theta}$ is as yet unspecified. Clearly, subtracting this equation from the equation of motion (3.2) and using Eq. (2.5) we obtain

$$\frac{\partial \vec{u}}{\partial t} = \nabla^2 \vec{u} - \vec{H}[\vec{\sigma} + \vec{u}] + \vec{H}[\vec{\sigma}] - \vec{\Theta}. \quad (3.4)$$

To this point things are quite general since we have not specified $\vec{\Theta}$. A key point is that $\vec{\Theta}$ must be chosen such that \vec{u} does indeed represent a fluctuation. This means that in the scaling regime we can treat \vec{u} as small and keep only leading powers of \vec{u} in the equations of motion for $\vec{\sigma}$ and \vec{u} . Equation (3.4), to leading order, is then given by

$$\frac{\partial u_i}{\partial t} = \nabla^2 u_i - W_{ij}[\vec{\sigma}] u_j - \Theta_i, \quad (3.5)$$

where a sum over the index j is assumed.

We now assume that $\vec{\Theta}$ is a function of \vec{m} only. This means that $\vec{\sigma}$ satisfies a closed equation, while \vec{u} is slaved by \vec{m} . We will choose the form for $\vec{\Theta}$ so that the correlation function f for \vec{m} is analytic for short-scaled distances. As we shall see this is a rather constrained process.

IV. ANALYSIS OF THE $\vec{\sigma}$ DEGREES OF FREEDOM

A. Construction of $\vec{\Theta}$

If we set $\vec{\Theta}$ equal to zero in Eq. (3.3) we obtain the equation used previously to determine the $\vec{\sigma}$ correlations [4]. This choice decouples $\vec{\sigma}$ and \vec{u} . The equation for \vec{u} would then separate into a (massless [16]) diffusion equation for the transverse piece \vec{u}_T and an equation for the longitudinal piece u_L with a mass term $-q_0^2 u_L$. However, the equation for $\vec{\sigma}$ would necessarily lead to nonanalytic behavior in f at short-scaled distances and ultimately to an unphysical divergence in $\tilde{g}(x)$ at small x . We must choose $\vec{\Theta}$ so that $f(x)$ is analytic for small x . The form we can use for $\vec{\Theta}$ is determined by the following observations: (i) $\vec{\Theta}$ must be odd under $\vec{m} \rightarrow -\vec{m}$. (ii) $\vec{\Theta}$ must scale as $O(L^{-2})$ in the scaling regime if it is to compensate for the terms in the equation of motion which lead to the nonanalyticities in f . This will also allow us to treat \vec{u} as a fluctuation since it will imply $\vec{u} \sim L^{-2}$.

It is not easy to construct a variety of functions of \vec{m} which are independent and satisfy (i) and (ii). We propose the general form

$$\vec{\Theta} = \frac{\omega_0}{L^2(t)} \vec{\sigma} + \sum_{l=1}^{l_{max}} \omega_l [\nabla \vec{m}]^{2(l-1)} \nabla_m^2 \vec{\sigma}, \quad (4.1)$$

where $[\nabla \vec{m}]^2 = \sum_{i=1}^d \sum_{\alpha=1}^n [\partial_i m_\alpha]^2$ and all of the ω_l , $l \geq 0$ are assumed to be of $O(1)$. One can think of including other quantities like $(\psi_0^2 - \vec{\sigma}^2) \vec{\sigma}$ but these, in the scaling regime, are equivalent to $\nabla_m^2 \vec{\sigma}$. It is interesting to note, using the definition (2.5) for $\nabla_m^2 \vec{\sigma}$, that $\vec{\Theta}$ is longitudinal.

For the purposes of this paper we will only consider constructing $f(x)$ to be analytic through terms of $O(x^4)$. To satisfy this requirement it is sufficient to set $l_{max} = 2$ in Eq. (4.1). The equation for $\vec{\sigma}$ (3.3) is then of the form

$$\vec{B} = \vec{0}, \quad (4.2)$$

where we define, for later convenience,

$$\vec{B} \equiv \partial_t \vec{\sigma} - \nabla^2 \vec{\sigma} + \nabla_m^2 \vec{\sigma} - \frac{\omega_0}{L^2(t)} \vec{\sigma} - \omega_1 \nabla_m^2 \vec{\sigma} - \omega_2 [\nabla \vec{m}]^2 \nabla_m^2 \vec{\sigma}. \quad (4.3)$$

B. The Gaussian approximation

To complete the definition of the model one must specify the form of the probability distribution for the auxiliary field

\vec{m} . Forcing $\vec{\sigma}$ to satisfy the exact equation of motion (4.2) is tantamount to solving the problem exactly, and will determine a probability distribution for \vec{m} which is complicated and extremely difficult for purposes of computation. Progress can be made if one imposes the weaker constraint

$$\langle \vec{B}(1) \cdot \vec{\sigma}(2) \rangle = 0. \quad (4.4)$$

This equation allows one to insure that $\vec{B}(1)$ is reasonably small at late times but gives one the flexibility to choose a suitable probability distribution. The simplest choice is a Gaussian probability distribution for \vec{m} with the correlation function $C_0(12)$ explicitly defined through

$$\langle m_i(1)m_j(2) \rangle = \delta_{ij} C_0(12). \quad (4.5)$$

The system is assumed to be statistically isotropic and homogeneous so $C_0(12)$ is invariant under interchange of its spatial indices. For future reference we also define the one-point correlation function

$$S_0(1) = C_0(11) \quad (4.6)$$

and the normalized correlation function

$$f(12) = \frac{C_0(12)}{\bar{S}_0(12)}, \quad (4.7)$$

with $\bar{S}_0(12) = \sqrt{S_0(1)S_0(2)}$. As discussed above it is expected that both C_0 and S_0 grow as L^2 at late times. The Gaussian approximation, which has been successful in describing the correlations in these systems, forms the basis of almost all present analytical treatments of phase-ordering problems [2,4]. Efforts to go beyond to Gaussian approximations are defined in [17,18].

The functional dependence of $C(12)$ and $\tilde{g}(x)$ on f can be derived without reference to the dynamics contained in Eq. (4.4). Using Eqs. (2.4), (2.6), and (2.8) $C(12)$ can be written to leading order in $1/L$

$$C(12) = \psi_0^2 \langle \hat{m}(1) \cdot \hat{m}(2) \rangle = \psi_0^2 \mathcal{F}(12). \quad (4.8)$$

Assuming Gaussian statistics for \vec{m} we obtain [4,19]

$$\mathcal{F}(12) = \frac{nf(12)}{2\pi} B^2 \left(\frac{1}{2}, \frac{n+1}{2} \right) F \left(\frac{1}{2}, \frac{1}{2}; \frac{n+2}{2}; f^2(12) \right), \quad (4.9)$$

where B is the beta function and F is the hypergeometric function. Within the Gaussian theory, $\tilde{g}(x)$ is given by [5]

$$\tilde{g}(x) = n! \left[\frac{h}{x} \right]^{n-1} \frac{\partial h}{\partial x}, \quad (4.10)$$

with $h = -\gamma f' / 2\pi$ and $\gamma = 1/\sqrt{1-f^2}$. The defect density is given by

$$n_0(t) = \frac{n!}{2^n \pi^{n/2} \Gamma(1+n/2)} \left[\frac{S_0^{(2)}}{nS_0(t)} \right]^{n/2} \quad (4.11)$$

with

$$S_0^{(2)} = \frac{1}{n} \langle [\nabla \vec{m}]^2 \rangle. \quad (4.12)$$

C. Order-parameter correlations

With the specification of the probability distribution for \vec{m} the constraint (4.4) leads to an equation that allows one to compute correlations in the order parameter $\vec{\sigma}$. The quantity $S_0^{(2)}$ (4.12), which will later appear in the definition of the length scale and in the formula for the autocorrelation exponent λ , is determined through condition (4.4), with $2=1$

$$\begin{aligned} & \frac{1}{2} \partial_{t_1} \langle \vec{\sigma}^2(1) \rangle - \langle \vec{\sigma}(1) \cdot \nabla_1^2 \vec{\sigma}(1) \rangle - \frac{\omega_0}{L^2(t_1)} \langle \vec{\sigma}^2(1) \rangle \\ & + (1-\omega_1) \langle \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(1) \rangle \\ & - \omega_2 \langle \vec{\sigma}(1) \cdot [\nabla_1 \vec{m}(1)]^2 \nabla_m^2 \vec{\sigma}(1) \rangle = 0. \end{aligned} \quad (4.13)$$

Equation (4.13) can be simplified by using the following identities, true for Gaussian averages,

$$\begin{aligned} \langle \vec{\sigma}(1) \cdot \nabla_1^2 \vec{\sigma}(1) \rangle &= -S_0^{(2)} \langle \nabla_m \cdot [\vec{\sigma}(1) \cdot \nabla_m \vec{\sigma}(1)] \rangle \\ &+ S_0^{(2)} \langle \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(1) \rangle, \end{aligned} \quad (4.14)$$

$$\langle \vec{\sigma}(1) \cdot [\nabla_1 \vec{m}(1)]^2 \nabla_m^2 \vec{\sigma}(1) \rangle = n S_0^{(2)} \langle \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(1) \rangle, \quad (4.15)$$

and observing that, at late times, the dominant term in Eq. (4.13) is

$$\begin{aligned} \langle \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(1) \rangle &= -\frac{\psi_0^2}{2S_0(1)} \ln S_0(1) \text{ for } n=2 \\ &= -\frac{\psi_0^2}{S_0(1)} \frac{n-1}{n-2} \text{ for } n>2. \end{aligned} \quad (4.16)$$

Evaluating Eq. (4.13) to leading order in $1/L$ one has

$$S_0^{(2)} = \frac{1-\omega_1}{1-2\mu(n-2)\omega_0/\pi(n-1)+n\omega_2}, \quad (4.17)$$

where we have defined the scaling length

$$L^2(t) = \frac{\pi S_0(t)}{2\mu S_0^{(2)}} = 4t. \quad (4.18)$$

Note that for $n=2$ the term with ω_0 does not appear in $S_0^{(2)}$ because it is dominated by the $O(L^{-2} \ln L)$ terms in Eq. (4.13). There is a further simplification of Eq. (4.17) for $n=2$ since later we will have to set $\omega_1 + 2S_0^{(2)}\omega_2 = 0$ to ensure that the correlations in \vec{u} remain finite. With this relation we have

$$S_0^{(2)} = 1 \quad \text{for } n=2, \quad (4.19)$$

which is the value for $S_0^{(2)}$ obtained previously for all n when $\vec{\Theta} = \vec{0}$.

Equation (4.4) directly determines the time evolution of the two-point order-parameter correlations and is given explicitly by

$$\left[\partial_{t_1} - \nabla_1^2 - \frac{\omega_0}{L^2(t_1)} \right] \langle \vec{\sigma}(1) \cdot \vec{\sigma}(2) \rangle + (1 - \omega_1) \langle \nabla_m^2 \vec{\sigma}(1) \cdot \vec{\sigma}(2) \rangle - \omega_2 \langle \nabla_m^2 \vec{\sigma}(1) [\nabla_1 \vec{m}(1)]^2 \cdot \vec{\sigma}(2) \rangle = 0. \quad (4.20)$$

Equation (4.20) can be reexpressed as an equation for the order-parameter correlation function \mathcal{F} (4.9) by means of the following identities:

$$\begin{aligned} & \langle \nabla_m^2 \vec{\sigma}(1) [\nabla_1 \vec{m}(1)]^2 \cdot \vec{\sigma}(2) \rangle \\ &= n S_0^{(2)} \langle \nabla_m^2 \vec{\sigma}(1) \cdot \vec{\sigma}(2) \rangle + [\nabla_1 C_0(12)]^2 \\ & \quad \times \langle \nabla_m^2 \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(2) \rangle, \end{aligned} \quad (4.21)$$

$$\langle \nabla_m^2 \vec{\sigma}(1) \cdot \vec{\sigma}(2) \rangle = \frac{-\psi_0^2}{S_0(1)} f \partial_f \mathcal{F}, \quad (4.22)$$

$$\langle \nabla_m^2 \vec{\sigma}(1) \cdot \nabla_m^2 \vec{\sigma}(2) \rangle = \frac{\psi_0^2}{S_0^2} (f \partial_f \mathcal{F} + f^2 \partial_f^2 \mathcal{F}), \quad (4.23)$$

where we use the shorthand notation $f = f(12)$, $\mathcal{F} = \mathcal{F}(12)$, and $\partial_f \mathcal{F} = \partial \mathcal{F} / \partial f$, etc. Equation (4.20) becomes

$$\left[\partial_{t_1} - \nabla_1^2 - \frac{\omega_0}{L^2(t_1)} \right] \mathcal{F} - \frac{1 - \omega_1 - n S_0^{(2)} \omega_2}{S_0(1)} f \partial_f \mathcal{F} - \omega_2 [\nabla_1 f]^2 [f \partial_f \mathcal{F} + f^2 \partial_f^2 \mathcal{F}] = 0, \quad (4.24)$$

which is the starting point for the evaluation of the two quantities of interest here: the autocorrelation exponent λ (1.3) and the late-time scaling form for \mathcal{F} .

For times $t_1 \gg t_2$ both \mathcal{F} and f are small. In this limit (4.24) becomes a linear equation for \mathcal{F} and, following the treatment in [4,11], with the definition (4.18), λ can be determined as

$$\lambda = d - \frac{\pi}{4\mu} \frac{1 - \omega_1 - n S_0^{(2)} \omega_2}{S_0^{(2)}} - \frac{\omega_0}{2}. \quad (4.25)$$

If one knows ω_0 , ω_1 , ω_2 , and μ one can determine λ . These quantities can be found from an analysis of the equal-time correlations, to which we now turn.

To examine the equal-time order-parameter correlations in the late-time scaling regime we set $t_1 = t_2 = t$ and write Eq. (4.24) in terms of the scaled distance x . To leading order in $1/L$ we have

$$\begin{aligned} & \vec{x} \cdot \nabla_x \mathcal{F} + \nabla_x^2 \mathcal{F} + \omega_0 \mathcal{F} + \frac{\pi}{2\mu} \frac{1 - \omega_1 - n S_0^{(2)} \omega_2}{S_0^{(2)}} f \partial_f \mathcal{F} \\ & + \omega_2 [\nabla_x f]^2 [f \partial_f \mathcal{F} + f^2 \partial_f^2 \mathcal{F}] = 0. \end{aligned} \quad (4.26)$$

The calculation of the scaling form for \mathcal{F} reduces to the solution of the nonlinear eigenvalue problem (4.26) with the eigenvalue μ . The eigenvalue is selected by finding numerically the solution of Eq. (4.26) which satisfies the analyti-

cally determined boundary behavior at both large and small x . The new aspect to the problem is the presence of the unknowns ω_0 , ω_1 , and ω_2 , a consequence of the incorporation of fluctuations into the model. For $n=2$ these constants play the role of counterterms that cancel out the small- x nonanalyticities in the normalized auxiliary field correlation function f . This procedure fixes ω_0 , ω_1 , and ω_2 in terms of μ and d .

For large x both \mathcal{F} and f are small and Eq. (4.26) can be linearized. In this regime the solution to Eq. (4.26) is

$$\mathcal{F} \sim x^{d-2\lambda} e^{-x^2/2}. \quad (4.27)$$

The result for the exponent $d-2\lambda$ appears to be robust. Until now, we have derived results valid for arbitrary $n > 1$. However, the primary goal of this paper is to examine the $O(2)$ model, where there are known qualitative discrepancies with simulation data. With this in mind, we now examine the small- x behavior of the scaling equation (4.26) for the case $n=2$. For small x (4.26) admits the following general expansion for f :

$$\begin{aligned} f = & 1 + f_2 x^2 \left\{ 1 + \frac{K_2}{\ln x} \left[1 + O\left(\frac{1}{\ln x}\right) \right] \right\} \\ & + f_4 x^4 \left\{ 1 + \frac{K_4}{\ln x} \left[1 + O\left(\frac{1}{\ln x}\right) \right] \right\} + O(x^6). \end{aligned} \quad (4.28)$$

Nonanalyticities appear as a result of the nonzero K_2 and K_4 coefficients multiplying factors of $1/\ln x$. The nonzero K_2 coefficient is particularly important since it is responsible for the divergence of the defect-defect correlation function at small x .

The coefficients of the expansion (4.28) can be determined by examining (4.26) order by order at small x . Balancing terms at $O(\ln x)$ gives

$$f_2 = -\frac{\pi}{4\mu d}. \quad (4.29)$$

This relation between f_2 and μ is the same one that was found in the original theory [4]. This equivalence is a consequence of the simplifications mentioned previously Eq. (4.19) that occur for $n=2$. At $O(1)$ we have an equation relating ω_0 , ω_2 , and K_2

$$\omega_0 = 2f_2(1 + dK_2 + \omega_2). \quad (4.30)$$

As discussed above, the constant $\omega_1 = -2\omega_2$ for $n=2$. If we work with $\omega_0 = \omega_1 = \omega_2 = 0$ then Eq. (4.30) implies $K_2 = -1/d$. This is simply the Gaussian model examined previously [4], whose nonzero value for K_2 results in the divergent small- x behavior of the defect-defect correlation function $\tilde{g}(x)$.

Now, however, we can insist that, at $O(x^2)$, f is analytic and enforce $K_2 = 0$. This choice produces the following relation between ω_0 and ω_2 :

$$\omega_0 = 2f_2(1 + \omega_2). \quad (4.31)$$

The small- x divergence in the defect-defect correlation function is now eliminated. The leading correction to the $O(x^2)$ term in f is then the $O(x^4)$ term with a coefficient

$$f_4 = \frac{d+3+\omega_2}{2(d+2)}(f_2)^2 - \frac{f_2(2+\omega_0)}{4(d+2)}. \quad (4.32)$$

We can go further and insist that f is analytic at small x up to $O(x^4)$. Enforcing $K_4=0$ allows one to arrive at a complicated expression for ω_2 in terms of μ and d only.

In Sec. VI we will consider enforcing $K_2=0$ through various choices for the ω_l and we will numerically solve the associated eigenvalue problem. Before doing this though we complete our discussion of the theory by examining in detail the correlations of the fluctuations and their relationship to the order-parameter correlations. In the process, we will establish the important constraint on the ω_l parameters discussed earlier.

V. ANALYSIS OF FLUCTUATION CORRELATIONS

In addition to order-parameter correlations the theory also completely describes correlations in the fluctuation field \vec{u} . There are two types of equal-time fluctuation correlations that are of interest to us. The first describes cross correlations between the $\vec{\sigma}$ and \vec{u} fields and is defined as

$$C_{u0}(12) = \langle \vec{u}(\mathbf{R}_1, t) \cdot \vec{\sigma}(\mathbf{R}_2, t) \rangle. \quad (5.1)$$

The second describes correlations of the fluctuation field with itself and is given by

$$\delta_{ij} C_{uu}(12) = \langle u_i(\mathbf{R}_1, t) u_j(\mathbf{R}_2, t) \rangle. \quad (5.2)$$

As we will see later these quantities are closely related in the scaling regime. One can deduce equations of motion for both C_{u0} and C_{uu} by using the equations of motion (3.3) and (3.5) for $\vec{\sigma}$ and \vec{u} . For equal times one has

$$\begin{aligned} \frac{\partial}{\partial t} C_{u0}(12) &= \nabla_1^2 C_{u0}(12) - \langle W_{ij}(1) u_j(1) \sigma_i(2) \rangle - C_{\Theta 0}(12) \\ &+ \nabla_2^2 C_{u0}(12) - C_{u2}(12) + C_{u\Theta}(12) \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} C_{uu}(12) &= \nabla_1^2 C_{uu}(12) - \frac{1}{n} \langle W_{ij}(1) u_j(1) u_i(2) \rangle \\ &- \frac{1}{n} C_{\Theta u}(12), \end{aligned} \quad (5.4)$$

where in the last equation we have used the translation invariance in space to combine two equivalent terms. We have also defined

$$C_{\Theta 0}(12) = \langle \vec{\Theta}(1) \cdot \vec{\sigma}(2) \rangle, \quad (5.5)$$

$$C_{u2}(12) = \langle \vec{u}(1) \cdot \nabla_m^2 \vec{\sigma}(2) \rangle, \quad (5.6)$$

$$C_{u\Theta}(12) = \langle \vec{u}(1) \cdot \vec{\Theta}(2) \rangle. \quad (5.7)$$

We can solve Eqs. (5.3) and (5.4) to determine the correlations for the fluctuation field \vec{u} if we make the additional assumption that \vec{u} is also a Gaussian field. In particular, we assume that \vec{m} and \vec{u} are coupled Gaussian fields satisfying

$$\begin{aligned} \langle m_i(1) \mathcal{G}(\vec{m}, \vec{u}) \rangle &= C_0(13) \left\langle \frac{\delta}{\delta m_i(3)} \mathcal{G}(\vec{m}, \vec{u}) \right\rangle \\ &+ C_{mu}(13) \left\langle \frac{\delta}{\delta u_i(3)} \mathcal{G}(\vec{m}, \vec{u}) \right\rangle, \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \langle u_i(1) \mathcal{G}(\vec{m}, \vec{u}) \rangle &= C_{um}(13) \left\langle \frac{\delta}{\delta m_i(3)} \mathcal{G}(\vec{m}, \vec{u}) \right\rangle \\ &+ C_{uu}(13) \left\langle \frac{\delta}{\delta u_i(3)} \mathcal{G}(\vec{m}, \vec{u}) \right\rangle, \end{aligned} \quad (5.9)$$

where \mathcal{G} is a general function of \vec{m} and \vec{u} , C_{um} is defined as

$$\delta_{ij} C_{um}(12) = \langle u_i(1) m_j(2) \rangle, \quad (5.10)$$

and integrations over \mathbf{R}_3 and t_3 are implied. Using these identities all correlation functions depending on \vec{m} and \vec{u} are determined in terms of C_0 , C_{um} , and C_{uu} . Thus, we will see that Eqs. (5.3) and (5.4) can be expressed in terms of C_{u0} , C_{uu} and averages over functions of \vec{m} alone. This is the first step in determining Eqs. (5.1) and (5.2). The second step is to evaluate the averages over \vec{m} , which can all be expressed in terms of C_0 , a quantity known from our analysis in Sec. IV. The final step is to analyze the equations resulting from Eqs. (5.3) and (5.4) in the late-time scaling regime and extract the scaling functions.

We begin by expressing all of the correlation functions involving a power of \vec{u} which appear in Eqs. (5.3) and (5.4) in terms of $C_{u0}(12)$, $C_{uu}(12)$, and averages over \vec{m} . Using the identity (5.9) we have

$$C_{u0}(12) = C_{um}(12) M_1, \quad (5.11)$$

$$C_{u2}(12) = C_{um}(12) M_3, \quad (5.12)$$

$$C_{u\Theta}(12) = C_{um}(12) \Omega_1, \quad (5.13)$$

$$\langle W_{ij}(1) u_j(1) \sigma_i(2) \rangle = C_{um}(11) W_1(12) + C_{um}(12) W_2(12), \quad (5.14)$$

$$\langle W_{ij}(1) u_j(1) u_i(2) \rangle = C_{uu}(12) \Omega_2 + C_{um}(21) C_{um}(11) \Omega_3, \quad (5.15)$$

where we define

$$M_1 = \langle \nabla_m \cdot \vec{\sigma}(1) \rangle, \quad (5.16)$$

$$M_3 = \langle \nabla_m \cdot \nabla_m^2 \vec{\sigma}(1) \rangle, \quad (5.17)$$

$$\Omega_1 = \frac{\omega_0}{L^2} M_1 + \sum_{l=1}^{l_{max}} \omega_l \langle \nabla_m \cdot \nabla_m^2 \vec{\sigma}(1) [\nabla_1 \vec{m}(1)]^{2(l-1)} \rangle, \quad (5.18)$$

$$\Omega_2 = \langle W_{ii}(1) \rangle, \quad (5.19)$$

$$\Omega_3 = \langle \nabla_m^i \nabla_m^j W_{ij}(1) \rangle, \quad (5.20)$$

$$W_1(12) = \langle [\nabla_m^j W_{ij}(1)] \sigma_i(2) \rangle, \quad (5.21)$$

$$W_2(12) = \langle W_{ij}(1) \nabla_m^j \sigma_i(2) \rangle. \quad (5.22)$$

We see from Eq. (5.11) that C_{um} can be eliminated in favor of C_{u0} in Eqs. (5.12)–(5.15). In terms of these various auxiliary functions, Eqs. (5.3) and (5.4) become

$$\begin{aligned} \frac{\partial}{\partial t} C_{u0}(12) &= \nabla_1^2 C_{u0}(12) - C_{u0}(11) W_1(12) M_1^{-1} \\ &\quad - C_{u0}(12) W_2(12) M_1^{-1} - C_{\Theta 0}(12) \\ &\quad + \nabla_2^2 C_{u0}(12) - C_{u0}(12) M_3 M_1^{-1} \\ &\quad + C_{u0}(12) \Omega_1 M_1^{-1} \end{aligned} \quad (5.23)$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} C_{uu}(12) &= \nabla_1^2 C_{uu}(12) - \frac{1}{n} C_{uu}(12) \Omega_2 - \frac{1}{n} C_{u0}(12) \\ &\quad \times [C_{u0}(11) \Omega_3 M_1^{-2} + \Omega_1 M_1^{-1}]. \end{aligned} \quad (5.24)$$

The next step is to compute the averages over \vec{m} Eqs. (5.16)–(5.22) in the scaling regime. We begin with the *one-point* averages. Except for Ω_1 , which involves spatial gradients, these can all be evaluated using

$$\langle A[\vec{m}] \rangle = \int \frac{d^n x}{(2\pi S_0)^{n/2}} e^{-x^2/(2S_0)} A[x]. \quad (5.25)$$

It is straightforward to show that

$$M_1 = \psi_0 \left(\frac{2}{S_0(t)} \right)^{1/2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \quad (5.26)$$

and

$$M_3 = -\frac{1}{S_0(t)} M_1. \quad (5.27)$$

Turning to Ω_2 we see rather trivially from the form of the stability matrix given by Eq. (2.9) that

$$\Omega_2 = q_0^2 + \mathcal{O}(L^{-2}), \quad (5.28)$$

with a $\ln L$ multiplying the correction term for $n=2$. This quantity serves as a *mass* for the longitudinal fluctuations and dominates their determination. For Ω_3 a brief manipulation produces the simple result

$$\Omega_3 = (n-1) \frac{q_0^2}{S_0(t)}. \quad (5.29)$$

The last local quantity Ω_1 , involves averages over spatial derivatives. The key point in handling such quantities is that

$\langle m_k(1) \nabla_i m_j(1) \rangle = 0$. One can then rather easily derive the recursion relation, valid for $l > 0$,

$$\langle (\nabla \vec{m})^{2l} G \rangle = n S_0^{(2)} \left[1 + \frac{2(l-1)}{nd} \right] \langle (\nabla \vec{m})^{2(l-1)} G \rangle \quad (5.30)$$

for a general local function $G[\vec{m}]$. With this relation we can evaluate Ω_1 , up to the ω_2 term

$$\Omega_1 = M_1 \left[\frac{\omega_0}{L^2(t)} - \frac{\omega_1}{S_0(t)} - \frac{n S_0^{(2)} \omega_2}{S_0(t)} \right]. \quad (5.31)$$

Turning to the *two-point* quantities W_1 and W_2 it is easy to show, using the symmetry properties of the order parameter, that in the scaling regime these reduce to

$$W_1(12) = (n-1) q_0^2 \psi_0 \left\langle \frac{\hat{\sigma}(1) \cdot \hat{\sigma}(2)}{m(1)} \right\rangle \quad (5.32)$$

and

$$W_2(12) = q_0^2 \psi_0 \left\langle \frac{1}{m(2)} \{1 - [\hat{\sigma}(1) \cdot \hat{\sigma}(2)]^2\} \right\rangle. \quad (5.33)$$

These are new averages to be evaluated. Hereafter, we shall work exclusively with $n=2$. In this case $W_1(12)$ and $W_2(12)$ can be evaluated as

$$W_1(12) = q_0^2 \psi_0 \left(\frac{\pi}{2S_0} \right)^{1/2} \frac{1}{f} (1 - \sqrt{1-f^2}), \quad (5.34)$$

while

$$W_2(12) = q_0^2 \psi_0 \left(\frac{\pi}{2S_0} \right)^{1/2} \frac{\sqrt{1-f^2}}{1 + \sqrt{1-f^2}}. \quad (5.35)$$

We are now in a position to evaluate C_{u0} and C_{uu} in the scaling regime, and to relate them to \mathcal{F} through $C_{\Theta 0}$. From the definition (4.1) of $\vec{\Theta}$ we see that the scaling *Ansatz* for $C_{\Theta 0}$ should have the form

$$C_{\Theta 0} = \frac{\psi_0^2}{L^2} F_{\Theta}(x), \quad (5.36)$$

where, after explicit evaluation,

$$\begin{aligned} F_{\Theta} &= \omega_0 \mathcal{F} - \frac{\pi}{2\mu S_0^{(2)}} (\omega_1 + 2S_0^{(2)} \omega_2) f \partial_f \mathcal{F} \\ &\quad + \omega_2 [\nabla_x f]^2 [f \partial_f \mathcal{F} + f^2 \partial_f^2 \mathcal{F}]. \end{aligned} \quad (5.37)$$

Looking at the determining equations (5.23) and (5.24) we easily see that, as a consequence of Eq. (5.36), we must take $u \sim L^{-2}$ to leading order. We therefore write

$$C_{u0}(12) = \frac{\psi_0^2}{L^2} F_u(x) \quad (5.38)$$

and

$$C_{uu}(12) = \frac{\psi_0^2}{L^4} F_{uu}(x). \quad (5.39)$$

With these *Ansätze* (5.23) can be written as

$$F_u(0) \frac{1}{f} (1 - \sqrt{1-f^2}) + F_u(x) \frac{\sqrt{1-f^2}}{1 + \sqrt{1-f^2}} = -\frac{1}{q_0^2} F_\Theta(x) \quad (5.40)$$

to leading order, while Eq. (5.24) becomes

$$\begin{aligned} F_{uu}(x) + \frac{2}{\pi} F_u(0) F_u(x) \\ = -\frac{1}{q_0^2} \left[\omega_0 - \frac{\pi}{2\mu S_0^{(2)}} (\omega_1 + 2S_0^{(2)} \omega_2) \right] F_u(x). \end{aligned} \quad (5.41)$$

We see that the quantity $F_u(0)$ enters into these equations. If this quantity is to be finite then we see that $F_\Theta(x)$ cannot blow up as $x \rightarrow 0$. Since $f \partial_f \mathcal{F}$ does blow up as $x \rightarrow 0$ we must choose

$$\omega_1 + 2S_0^{(2)} \omega_2 = 0, \quad (5.42)$$

which fixes ω_1 in terms of ω_2 and tells us, using Eq. (4.17), that $S_0^{(2)} = 1$ for $n=2$, even in the presence of these perturbations. We then find that

$$F_u(0) = \frac{2\omega_2 f_2 - \omega_0}{q_0^2} = -\frac{2f_2(1 + dK_2)}{q_0^2}, \quad (5.43)$$

where Eq. (4.30) has been used. We then have the final results

$$\begin{aligned} F_u(x) = -\frac{\gamma}{q_0^2} \{ \omega_0 [(1 + \sqrt{1-f^2}) \mathcal{F} - f] + 2\omega_2 f_2 f \\ + \omega_2 (1 + \sqrt{1-f^2}) [\nabla_x f(x)]^2 [f \partial_f \mathcal{F} + f^2 \delta_f^2 \mathcal{F}] \} \end{aligned} \quad (5.44)$$

and

$$F_{uu}(x) = -\frac{1}{q_0^2} \left[\omega_0 + \frac{2q_0^2}{\pi} F_u(0) \right] F_u(x). \quad (5.45)$$

Inspection of Eqs. (5.43)–(5.45) show that in the original theory [4] $F_u(x) = F_{uu}(x) = 0$, as expected. From the definitions (5.2) and (5.39) we must have $F_{uu}(0) \geq 0$. In the theory with only $\omega_0 \neq 0$ we have

$$F_{uu}(0) = \frac{\omega_0^2}{q_0^4} \left(1 - \frac{2}{\pi} \right) \quad (5.46)$$

which is positive. However, if $\omega_0 = 0$ Eq. (5.45) implies

$$F_{uu}(0) = -\frac{2}{\pi} [F_u(0)]^2 \quad (5.47)$$

which is negative. Thus within the $l_{max}=2$ approximation it is necessary to have $\omega_0 \neq 0$ in order to have a physical

TABLE I. Values for the eigenvalue μ from the various theories (in all theories $\omega_1 = -2\omega_2$).

	$\omega_0 = \omega_2 = 0$	$\omega_0 \neq 0, \omega_2 = 0$	$\omega_0 = 0, \omega_2 = -1$	$\omega_0, \omega_2 \neq 0$
$d=2$	0.948 58	0.537 21	0.760 33	0.428 63
$d=3$	0.568 37	0.389 31	0.514 34	0.325 44

theory. For more general $\vec{\Theta}$, one must look to the numerical solution of Eq. (4.26) to answer the question of the sign of $F_{uu}(0)$.

Equations (5.44) and (5.45) explicitly show how correlations in the \vec{u} field are slaved to those of the order parameter. The universality in Eqs. (5.44) and (5.45) is evident, up to the nonuniversal overall factor of $1/q_0^4$, which characterizes the flatness of the equilibrium minimum in the potential and sets the scale of the fluctuations.

VI. NUMERICAL ANALYSIS OF THE NONLINEAR EIGENVALUE PROBLEM

The eigenvalue problem posed by Eq. (4.26), subject to the boundary conditions at small and large x outlined above, has to be solved numerically. A fourth-order Runge-Kutta integrator is used to integrate Eq. (4.26) with initial conditions given by an analytic small- x expansion to $x=0.0001$. The eigenvalue μ is adjusted until the solution matches onto the Gaussian decay (4.27) at the largest distances. This is now a standard procedure and is essentially the same as that used in [4]. We examine the O(2) model in two and three spatial dimensions.

In the original theory [4] $\vec{\Theta} = \vec{0}$ and the selection of the eigenvalue depended on only the conditions outlined above. In that theory $K_2 = -1/d$, which lead to an unsatisfactory nonanalyticity in the small- x behavior of $f(x)$ and ultimately to an unphysical divergence in $\vec{g}(x)$ at small x . In the present theory we can choose $\vec{\Theta}$ so as to eliminate the leading nonanalyticities and remove the unphysical divergence. Each choice represents a separate eigenvalue problem. The simplest choice is to keep only the first term in $\vec{\Theta}$ by setting $\omega_0 = 2f_2$ and $\omega_l = 0$ for $l > 0$. For $d=2$ the solution to this eigenvalue problem has $\omega_0 = -1.4620 \dots$, while for $d=3$ one has $\omega_0 = -1.3450 \dots$. Another choice for $\vec{\Theta}$ is to eliminate the first term, but keep the next two by setting $\omega_0 = 0$, $\omega_1 = 2$, $\omega_2 = -1$ and $\omega_l = 0$ for $l > 2$. As mentioned in the preceding section, this choice has the unfortunate consequence of rendering $F_{uu}(0)$ negative. Both these theories have $K_2 = 0$ and $K_4 \neq 0$. If we require $f(x)$ to be analytic up to $O(x^4)$ we must choose all of ω_0 , ω_1 , and ω_2 to be nonzero, following the prescription outlined in Sec. IV to ensure that $K_2 = K_4 = 0$. The solution to this eigenvalue problem has $\omega_0 = -2.6004 \dots$, $\omega_2 = 0.419 14 \dots$ for $d=2$ and $\omega_0 = -2.3438 \dots$, $\omega_2 = 0.456 75 \dots$ for $d=3$.

Table I contains the eigenvalues μ obtained from these theories. The autocorrelation exponents are shown in Table II. The scaling forms for the order-parameter correlation functions \mathcal{F} from the various theories are compared in Fig. 1 for the O(2) model in two dimensions. The three dimensional

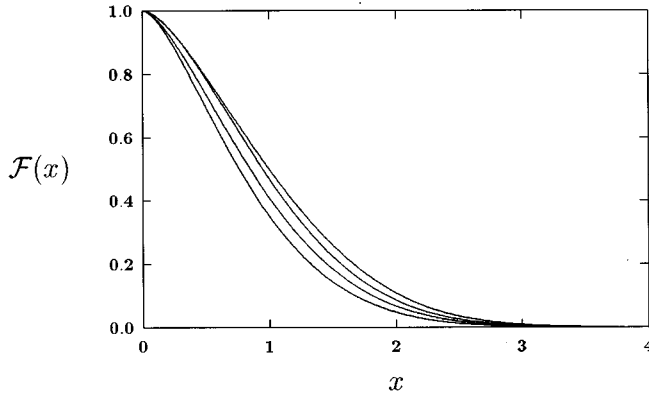


FIG. 1. Scaling function $\mathcal{F}(x)$ for the order-parameter correlations in two dimensions. From bottom to top, at $x=1$, the curves correspond to the theory with $\omega_0 \neq 0$ and $\omega_2 \neq 0$; the theory with only $\omega_0 \neq 0$; the unmodified theory [4]; the theory with $\omega_0 = 0$ and $\omega_2 \neq 0$. In all theories $\omega_1 = -2 \omega_2$.

results for \mathcal{F} are similar, and are not shown. The defect-defect correlation function $\tilde{g}(x)$ (4.10) is calculated from our numerically determined form for $f(x)$. The results for the correlations between vortices ($n=d=2$) obtained from the various theories are compared in Fig. 2. Simulation results [6], which are scaled to give the best fit to the original theory [4] at large x , are shown for comparison. Finally, the scaling function F_{uu} (5.45) is computed using our knowledge of \mathcal{F} and f . The results for the theory with $\omega_0 \neq 0$ and $\omega_2 \neq 0$ are shown in Fig. 3 for two and three spatial dimensions. For the theory with just $\omega_0 \neq 0$ the behavior of F_{uu} is similar to that shown.

VII. DISCUSSION

It is noteworthy that the scaling solution for we find here for \mathcal{F} retains the same qualitative features seen in [4]. The differences between the \mathcal{F} s can be attributed mainly to a rescaling of the length scale. Since there is always some

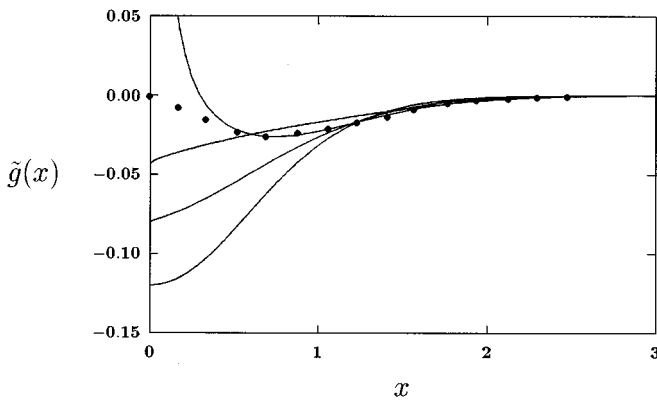


FIG. 2. Scaling function $\tilde{g}(x)$ for the defect-defect correlations in two dimensions. From bottom to top at $x=0$ the solid curves correspond to the theory with $\omega_0 \neq 0$ and $\omega_2 \neq 0$; the theory with only $\omega_0 \neq 0$; the theory with $\omega_0 = 0$ and $\omega_2 \neq 0$, the unmodified theory [4] (diverging). In all theories $\omega_1 = -2 \omega_2$. The dots represent the simulation results [6] for the two dimensional O(2) model.

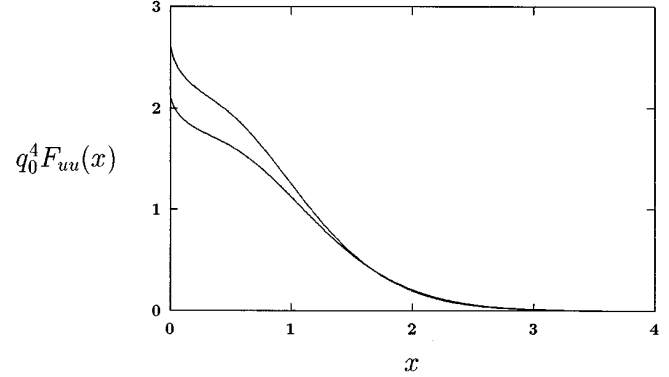


FIG. 3. Scaling function for the fluctuation correlations for the theory with $\omega_0 \neq 0$, $\omega_2 \neq 0$, and $\omega_1 = -2 \omega_2$. At $x=0$ the lower curve is the result for three dimensions and the upper curve is the result for two dimensions.

arbitrariness in the choice of length scale when comparing simulation data to theory, we expect that our new results for \mathcal{F} will fit the simulation data well after rescaling. To our knowledge, simulation results for the autocorrelation exponent λ of the O(2) model exist only for two spatial dimensions [12], where it is found that $\lambda = 1.171$. We see from Table II that the original theory [4] is already in excellent agreement with the simulations on this point. It is not surprising then that the modified theory makes worse predictions for λ than the original theory. The introduction of $\tilde{\Theta}$ was not expected to be in any sense a small perturbation. A trend which is counter to our expectations is that the discrepancy between the value for λ from simulations and the value obtained in the theory seems to increase as one includes more terms in $\tilde{\Theta}$. It may still turn out that the inclusion of higher order terms in l in $\tilde{\Theta}$ does lead to improvement. This appears to be a straightforward but tedious calculation. It is also interesting to note that, for the theory with $\omega_0 = 0$ and $\omega_2 \neq 0$, the prediction for λ violates a proposed [20] lower bound $\lambda > d/2$. Despite these problems with the quantitative values obtained for λ , there is qualitative improvement in $\tilde{g}(x)$ since the small- x divergence for $n=d=2$ seen in the original theory is removed. The value of $\tilde{g}(0)$ in all the modified theories is too low when compared with simulations and this point suggests that some fine-tuning of the theory is necessary. Finally, our results for F_{uu} obey the necessary condition $F_{uu}(0) > 0$ for two of our choices of $\tilde{\Theta}$. We also see that the strength of the fluctuations, characterized by $F_{uu}(0)$, increases in lower dimensions, as one might expect.

We see that the inclusion of fluctuations allows us to ren-

TABLE II. Values for the autocorrelation exponent λ from the various theories (in all theories $\omega_1 = -2 \omega_2$).

	$\omega_0 = \omega_2 = 0$	$\omega_0 \neq 0, \omega_2 = 0$	$\omega_0 = 0, \omega_2 = -1$	$\omega_0, \omega_2 \neq 0$
$d=2$	1.1720	1.2690	0.967 03	1.4678
$d=3$	1.6182	1.6550	1.4730	1.7585

der the correlation function f of the auxiliary field analytic and we have constructed such a solution up to $O(x^4)$. This, in turn, cures the short-distance divergence in \tilde{g} . The theory, at the Gaussian level, appears to be in better qualitative shape given our development here, however it is at the expense of proper quantitative agreement for the nonequilibrium exponent λ . There is evidence [18] that the inclusion of post-Gaussian corrections lowers the value of λ . Thus we hope that the tendency for the fluctuations to increase λ will be balanced by the introduction of post-Gaussian terms, resulting in a value for λ in reasonable agreement with simulations. We also hope that post-Gaussian corrections will reduce the magnitude of $\tilde{g}(0)$. It appears that the procedure we

introduce here leads to a qualitatively more consistent theory. However it is also clear that it is unlikely that one can have such a theory and quantitative estimates for exponents within the Gaussian approximation. One should proceed to look at post-Gaussian theories.

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